# The Perfectly Matched Layer Absorbing Boundary Condition for Maxwell's Equations

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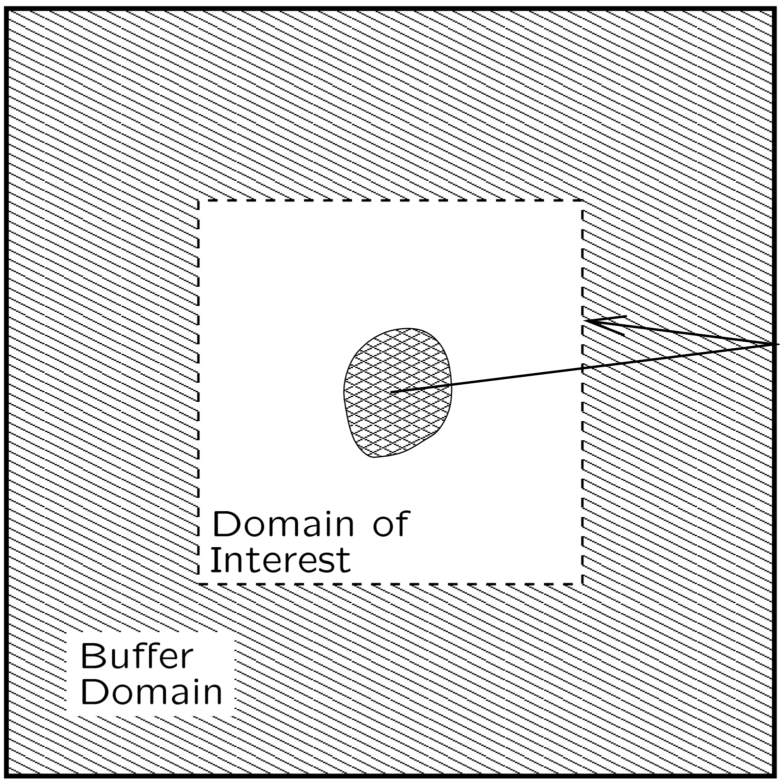
**Outline:**

1. Perfectly Matched Layer for Maxwell's Equations

2. Mimetic Difference Operators

3. Computational Results

## Absorbing Boundary Conditions

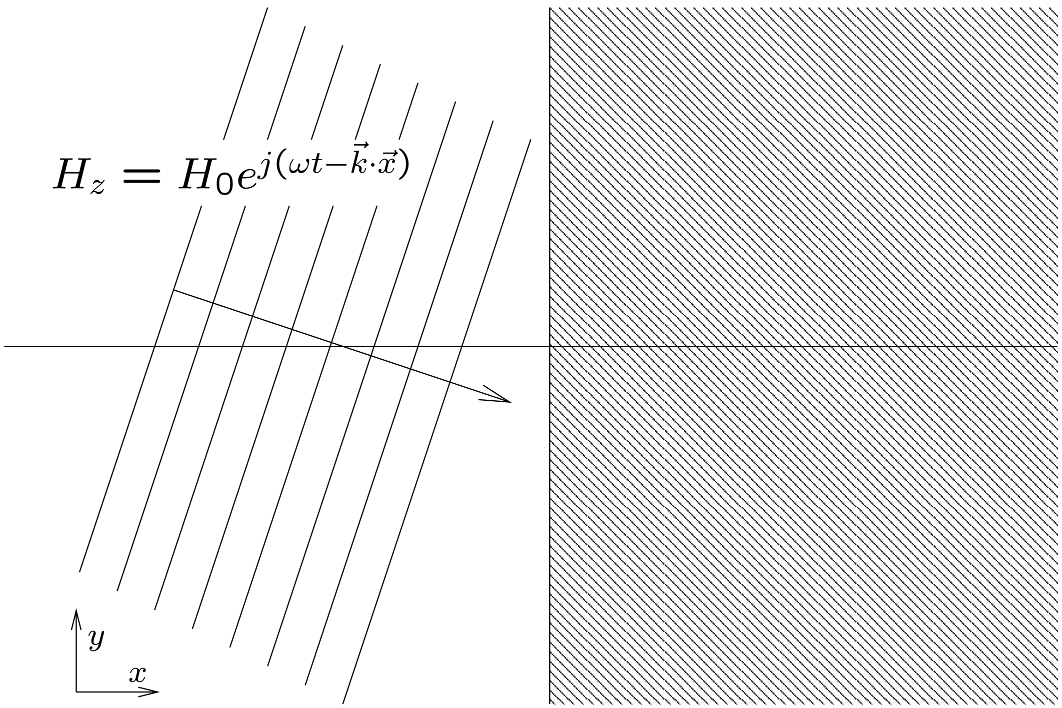



Problem: Duration of simulation limited by return of reflected waves from boundary.

Solution: Enlarge computational area, *or* reduce boundary reflections.

# The Perfectly Matched Layer

Berenger (1994) considered the problem of attaining perfect transmission of planar electromagnetic waves from dielectric media.




$$
\begin{aligned}
\epsilon \frac{\partial E}{\partial t} + \sigma E &= \nabla \times H \\
\mu \frac{\partial H}{\partial t} + \sigma^* H &= -\nabla \times E
\end{aligned}
$$

## Field Splitting (TE mode)

$$\left.\begin{aligned}
\epsilon\frac{\partial E_x}{\partial t} + \sigma E_x &= \frac{\partial H_z}{\partial y} \\
\epsilon\frac{\partial E_y}{\partial t} + \sigma E_y &= -\frac{\partial H_z}{\partial x} \\
\mu\frac{\partial H_z}{\partial t} + \sigma^* H_z &= \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}
\end{aligned}\right\} \Longrightarrow
\begin{aligned}
\epsilon\frac{\partial E_x}{\partial t} + \sigma_y E_x &= \frac{\partial H_z}{\partial y} \\
\epsilon\frac{\partial E_y}{\partial t} + \sigma_x E_y &= -\frac{\partial H_z}{\partial x} \\
\mu\frac{\partial H_{zy}}{\partial t} + \sigma_x^* H_{zx} &= -\frac{\partial E_y}{\partial x} \\
\mu\frac{\partial H_{zx}}{\partial t} + \sigma_y^* H_{zy} &= \frac{\partial E_x}{\partial y}
\end{aligned}$$

where $H_z = H_{zx} + H_{zy}$.

Note new parameters: $\sigma_x, \sigma_x^*, \sigma_y, \sigma_y^*$. These characterize the PML layer.

Free space is equivalent to a PML layer with $\sigma_x, \sigma_x^*, \sigma_y, \sigma_y^* = 0$.

Substitution yields:

$$H_z = H_0 e^{j\omega\left(t - \frac{\epsilon x \cos\theta + \epsilon y \sin\theta}{CG}\right)} e^{-\frac{\sigma_x \cos\theta}{\epsilon CG} x} e^{-\frac{\sigma_y \sin\theta}{\epsilon CG} y}$$

where

$$G = \sqrt{w_x \cos^2\theta + w_y \sin^2\theta}$$

$$w_x = \frac{1 + j\sigma_x/\omega\epsilon}{1 - j\sigma_x^*/\omega\mu}; \qquad w_y = \frac{1 + j\sigma_y/\omega\epsilon}{1 - j\sigma_y^*/\omega\mu}$$

Reflectionless transmission and attenuation occur when the *Impedance matching conditions* are observed:

$$\frac{\sigma_x}{\epsilon} = \frac{\sigma_x^*}{\mu} \quad \text{in } x \text{ direction}$$

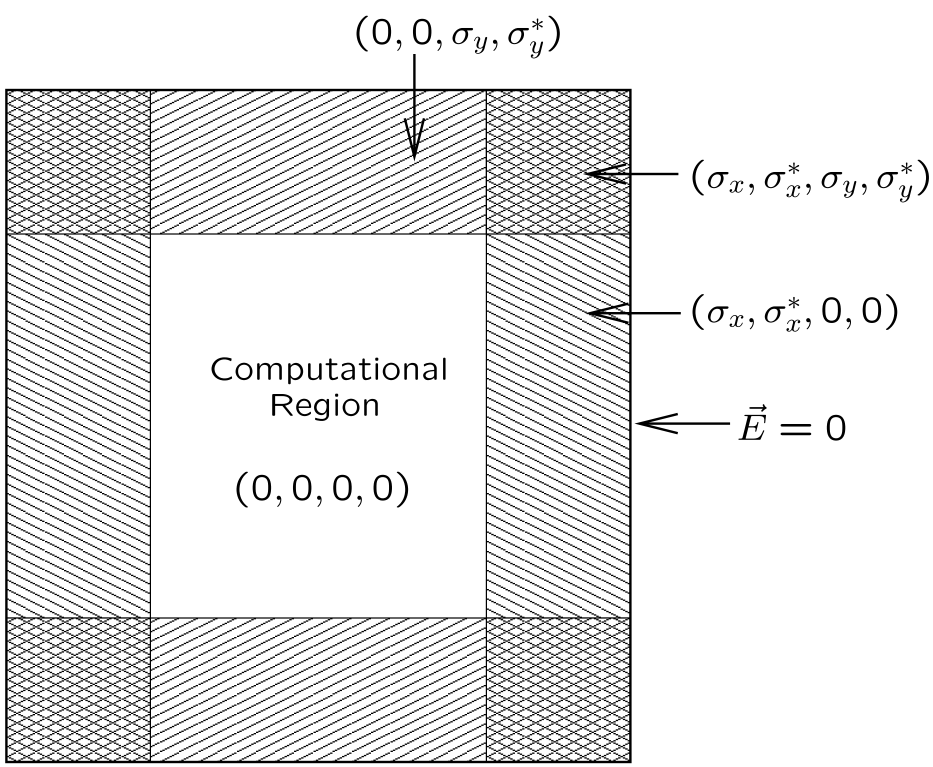
$$\frac{\sigma_y}{\epsilon} = \frac{\sigma_y^*}{\mu} \quad \text{in } y \text{ direction}$$

The perfect transmission of plane waves occurs *regardless of angle of incidence or frequency*.

By using both matching conditions, PMLs can be matched to free space, or to other PML layers.

## Split-field PML as an absorbing layer

Surround the computational region with finite depth PMLs.




Use impedance matching in both $x$ and $y$ in corner regions.

Terminate the PMLs with a simple BC, such as a perfect electrical conductor.

Advantages:

- Actual reflection magnitudes roughly $10^3$ times better than typical second and third order analytical ABCs.
- System remains in time domain.

Disadvantages:

- Non-physical modification of Maxwell's Equations.
- Extra variables introduced by field splitting

## Uniaxial Formulation

Allow material parameters to be diagonal tensors. In the frequency domain:

$$\begin{aligned}\nabla \times E &= -j\omega[\mu]H - [\sigma^*]H \\ \nabla \times H &= j\omega[\epsilon]E + [\sigma]E\end{aligned}$$

Impedance matching requires:

$$\frac{[\epsilon] + [\sigma]/j\omega}{\epsilon_0} = \frac{[\mu] + [\sigma^*]/j\omega}{\mu_0} = \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} = [\Lambda]$$

Consider a PML interface in the $yz$ plane.

Phase matching yields:

$$\begin{aligned} \sin\theta_i &= \sin\theta_r \\ \sqrt{bc}\sin\theta_t &= \sin\theta_i \end{aligned}$$

And the reflection coefficient for the TE/TM modes are:

$$R^{TE} = \frac{\cos\theta_i - \sqrt{\frac{b}{a}}\cos\theta_t}{\cos\theta_i + \sqrt{\frac{b}{a}}\cos\theta_t},$$

$$R^{TM} = \frac{\sqrt{\frac{b}{a}}\cos\theta_t - \cos\theta_i}{\cos\theta_i + \sqrt{\frac{b}{a}}\cos\theta_t}$$

The conditions $\sqrt{bc} = 1$ and $a = b$ yield $R^{TE} = R^{TM} = 0$.

The following choice of $\Lambda$:

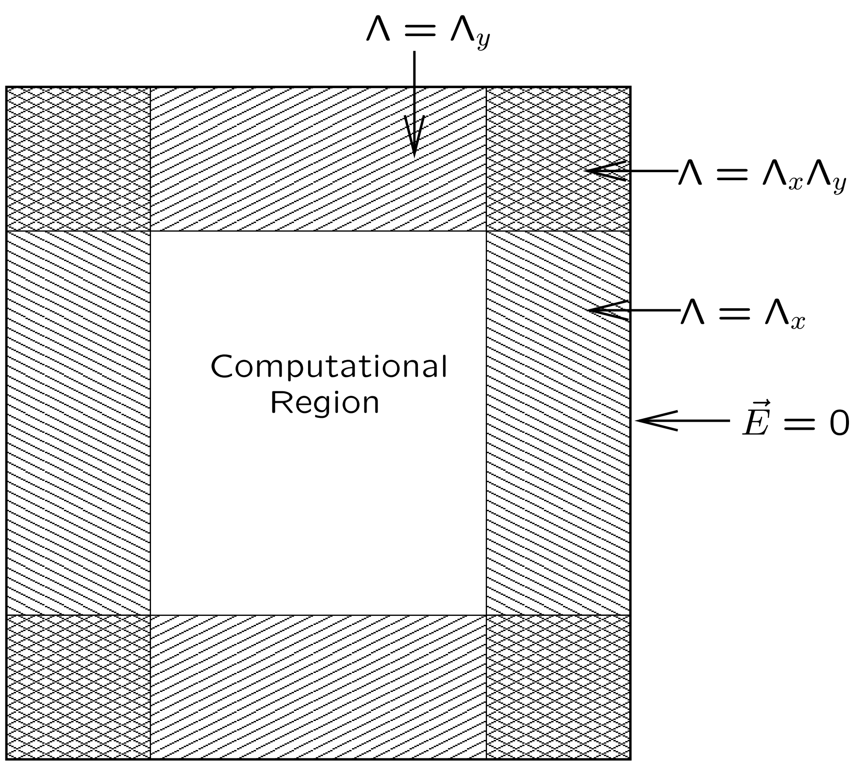
$$\frac{[\mu]}{\mu_0} = \frac{[\epsilon]}{\epsilon_0} = [\Lambda] = \begin{bmatrix} 1/a & & \\ & a & \\ & & a \end{bmatrix}$$

allows the transmission of traveling waves into the half space without reflection *regardless of the frequency or angle of incidence*.

If $a = 1 + \frac{\sigma}{j\omega}$ then the waves are attenuated with depth in the layer.

## Application of Perfectly Matched Layers to Simulations

Surround the computational region with PML layers of finite depth.




Composite $\Lambda$ in corner region is product.

Terminate PML with a simple BC, such as a perfect electrical conductor.

## Application of Perfectly Matched Layers to Simulations

Choose $\sigma$ in each layer to be zero near the surface and increase with depth in the layer. Powers of degree $3 \sim 4$ work well.

PML equations are all in the form $A = sB$ where $s = 1 + \sigma/j\omega$ where $A$ and $B$ are field components or intermediate variables.

Identify dependent variables in PML equations; convert back to time domain.

$$\begin{aligned} A &= (1 + \frac{\sigma}{j\omega})B \\ j\omega A &= (j\omega + \sigma)B \\ \frac{\partial A}{\partial t} &= \frac{\partial B}{\partial t} + \sigma B \end{aligned}$$

## PML Implementations

- FDTD. Yee scheme with staggered time stepping.

- FEM and spectral methods. Time and frequency domain.

- Versions of the PML native to other orthogonal coordinate systems.

- Higher order FD schemes.

## PML Variations

- Different PML parameters. (e.g. $s = 1 + \frac{\sigma}{\beta + j\omega}$)
- Different constitutive laws. (e.g. $D = [\Lambda]E + [M]H$)
- PMLs which match dispersive, non-linear, anisotropic media.
- Versions for acoustics, NLS.

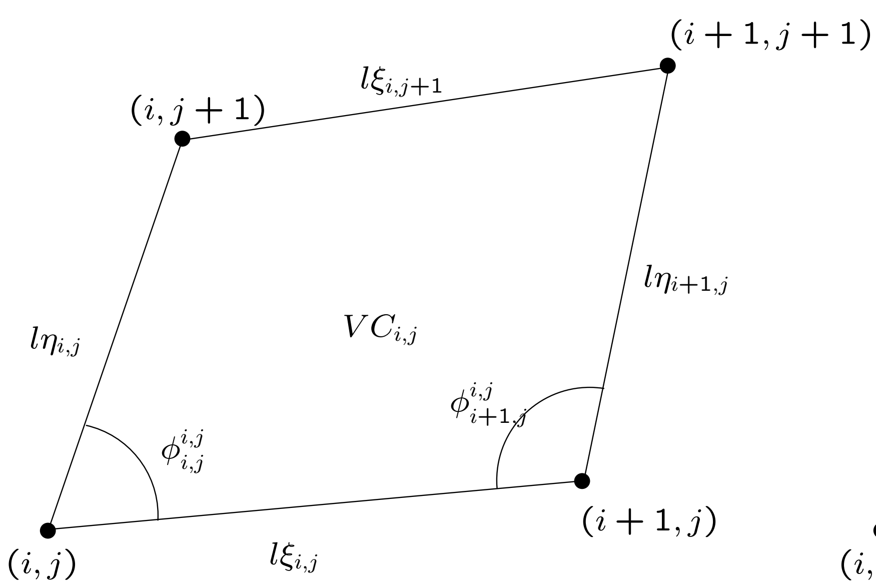
## Mimetic Difference Operators

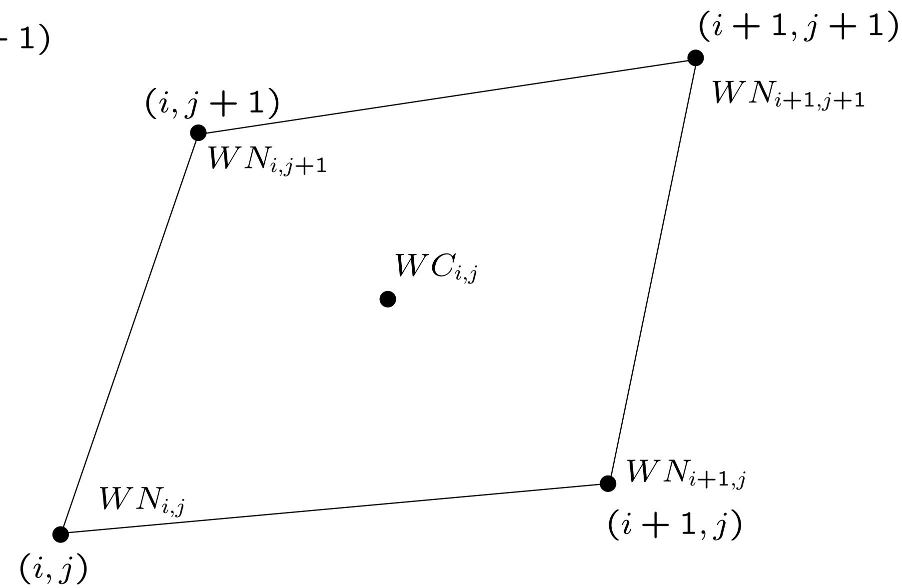
Discrete operators which *mimic* properties of continuous operators.

- Obey discrete forms of vector identities
- Eliminate spurious modes in solutions
- Enforce conservation laws
- Allow irregular (structured or unstructured) grids

# Discrete Scalar Spaces

Logically rectangular grids:

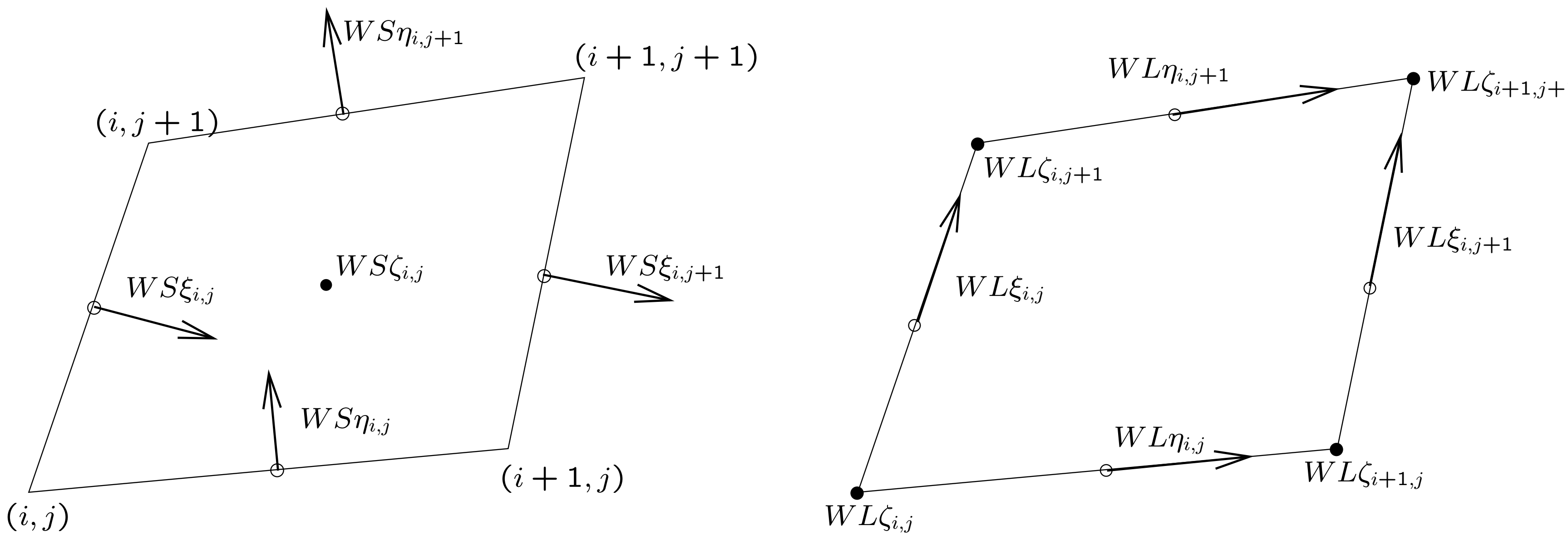



$HN$: Nodal values

$HC$: Cell-centered values

# Discrete Vector Spaces




$\mathcal{H}S$: Vectors normal to cell sides

$\mathcal{H}L$: Vectors tangent to cell sides

Natural Operators

**div** : $\mathcal{H}S \mapsto HC$
**grad** : $HN \mapsto \mathcal{H}L$
**curl** : $\mathcal{H}L \mapsto \mathcal{H}S$

$$\nabla \cdot \vec{W} = \lim_{|V| \to 0} \frac{1}{|V|} \iint_{\partial V} (\vec{W}, \hat{n}) dS$$

$$(\nabla \times \vec{W}, \hat{n}) = \lim_{|S| \to 0} \frac{1}{|S|} \oint_{\partial S} (\vec{W}, \hat{l}) ds$$

The natural operators satisfy the following identities:

**div** $\vec{A} = 0$ iff $\vec{A} =$ **curl** $\vec{B}$ where $\vec{A} \in \mathcal{H}S$ and $\vec{B} \in \mathcal{H}L$

**curl** $\vec{A} = 0$ iff $\vec{A} =$ **grad** $U$ where $\vec{A} \in \mathcal{H}L$ and $U \in HN$

Note that we can not form **lap** $\equiv$ **div grad** because the domains do not match.

## Adjoint Operators

$$\int_V u \nabla \cdot \vec{W} dV + \int_V (\vec{W}, \nabla u) dV = \oint_{\partial V} u(\vec{W}, \hat{n}) dS$$

$$\int_V (\vec{A}, \nabla \times \vec{B}) dV - \int_V (\vec{B}, \nabla \times \vec{A}) dV = \oint_{\partial V} (\hat{n}, \vec{A} \times \vec{B}) dS$$

Discrete versions of these identities are used to define adjoint operators.

These also satisfy the same vector identities as before.

$$\begin{array}{ccc}
\overline{\textbf{div}} : \mathcal{H}L \mapsto HN & \textbf{div}\ \overline{\textbf{grad}} : HC \to HC & \overline{\textbf{div}}\ \textbf{grad} : HN \to HN \\
\overline{\textbf{grad}} : HC \mapsto \mathcal{H}S & \textbf{curl}\ \overline{\textbf{curl}} : \mathcal{H}S \to \mathcal{H}S & \overline{\textbf{curl}}\ \textbf{curl} : \mathcal{H}L \to \mathcal{H}L \\
\overline{\textbf{curl}} : \mathcal{H}S \mapsto \mathcal{H}L & \textbf{grad}\ \overline{\textbf{div}} : \mathcal{H}L \to \mathcal{H}L & \overline{\textbf{grad}}\ \textbf{div} : \mathcal{H}S \to \mathcal{H}S
\end{array}$$

## Maxwell's Equations

$$\frac{\partial B}{\partial t} = -\nabla \times E \quad \nabla \cdot B = 0$$

$$\frac{\partial D}{\partial t} = \nabla \times H \quad \nabla \cdot D = 0$$

With constitutive laws $D = \epsilon E$, $B = \mu H$, these become:

$$\frac{\partial B}{\partial t} = -\nabla \times E$$

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon}\nabla \times \frac{1}{\mu} B$$

We assign the discrete functions $\vec{E} \in \mathcal{H}L$, $\vec{B} \in \mathcal{H}S$ and use the operators **curl**, $\overline{{}_{\epsilon}\mathbf{curl}_{\mu}}$

## Mimetic Discretization of Maxwell's Equations with PML

Change variables with $D = \epsilon E_p$, $B_p = \mu H$. Maxwell's equations become:

$$j\omega B = -\nabla \times E \quad E_p = \Lambda E$$

$$j\omega E_p = \frac{1}{\epsilon}\nabla \times \frac{1}{\mu}B_p \quad B_p = \Lambda^{-1}B$$

These can be converted back into the time domain and discretized as before. PML equations are of the form $A = sB$ where $s = 1 + \sigma/j\omega$.

Time stepping is done with a leapfrog scheme: $E^n, E_p^n$ and $B^{n+1/2}, B_p^{n+1/2}$.

## Implementation issues

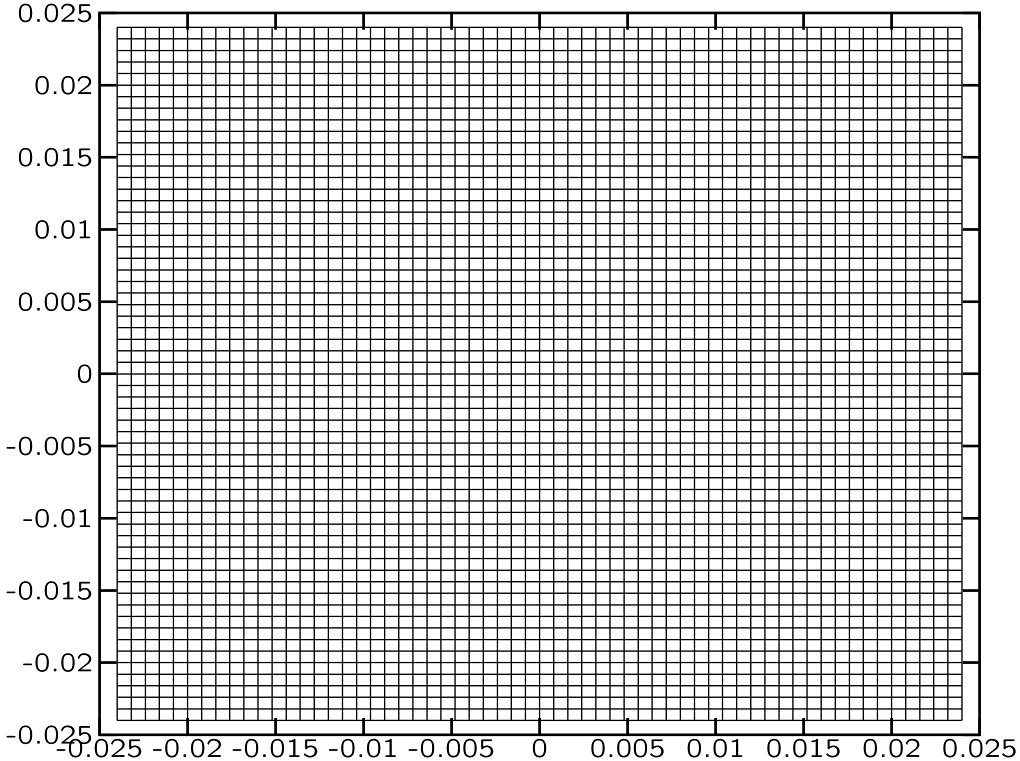
Note that the implementation avoids incorporating the PML tensors into the **curl** operator.

PML parameters considered functions of the variables $\xi, \eta$ interpolated between grid lines.

Since we're using the orthogonal form of the PML, the grid must be at least approximately orthogonal.

## Example Results

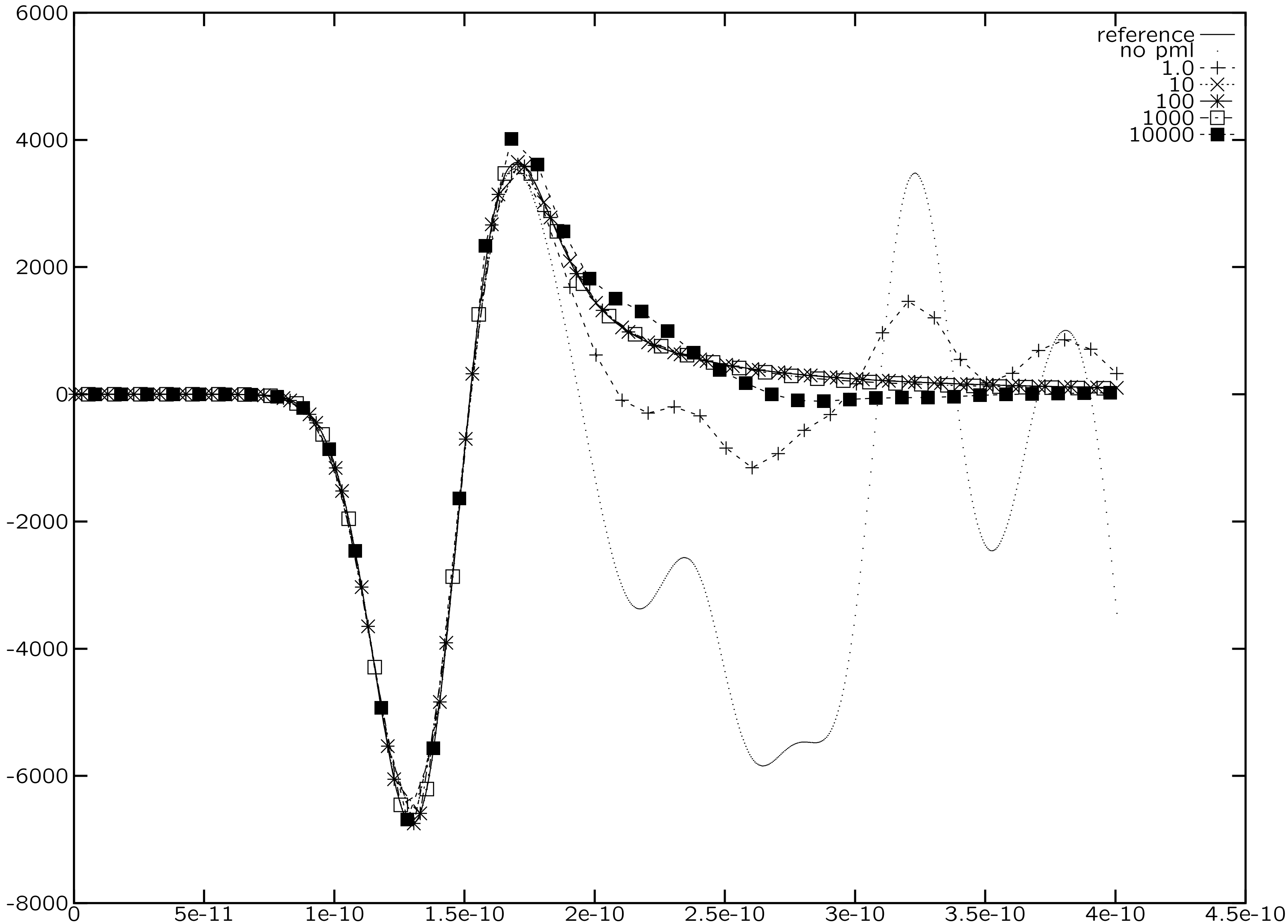
A current source is added to Maxwell's equations and a pulse generated in the center of the domain $[-0.024, 0.024] \times [-0.024, 0.024]$. The resulting $B_z$ is observed at $(-0.012, 0.012)$.




$60 \times 60$ grid.

PML width 10 cells.

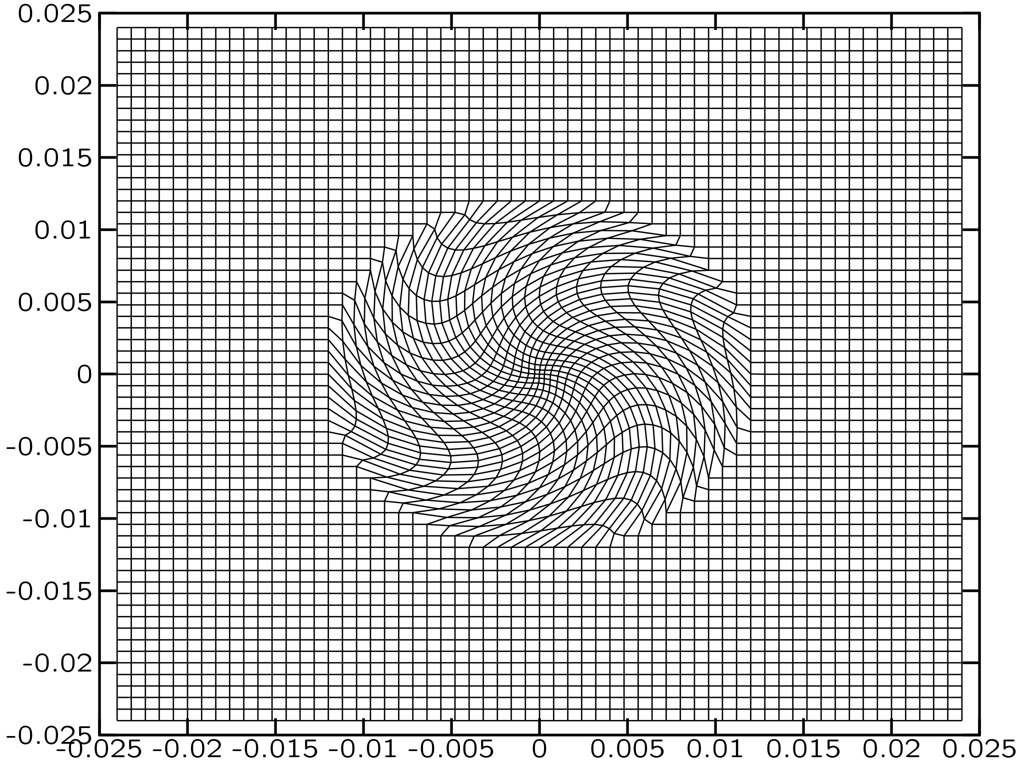
Various $\sigma_{\text{max}}$

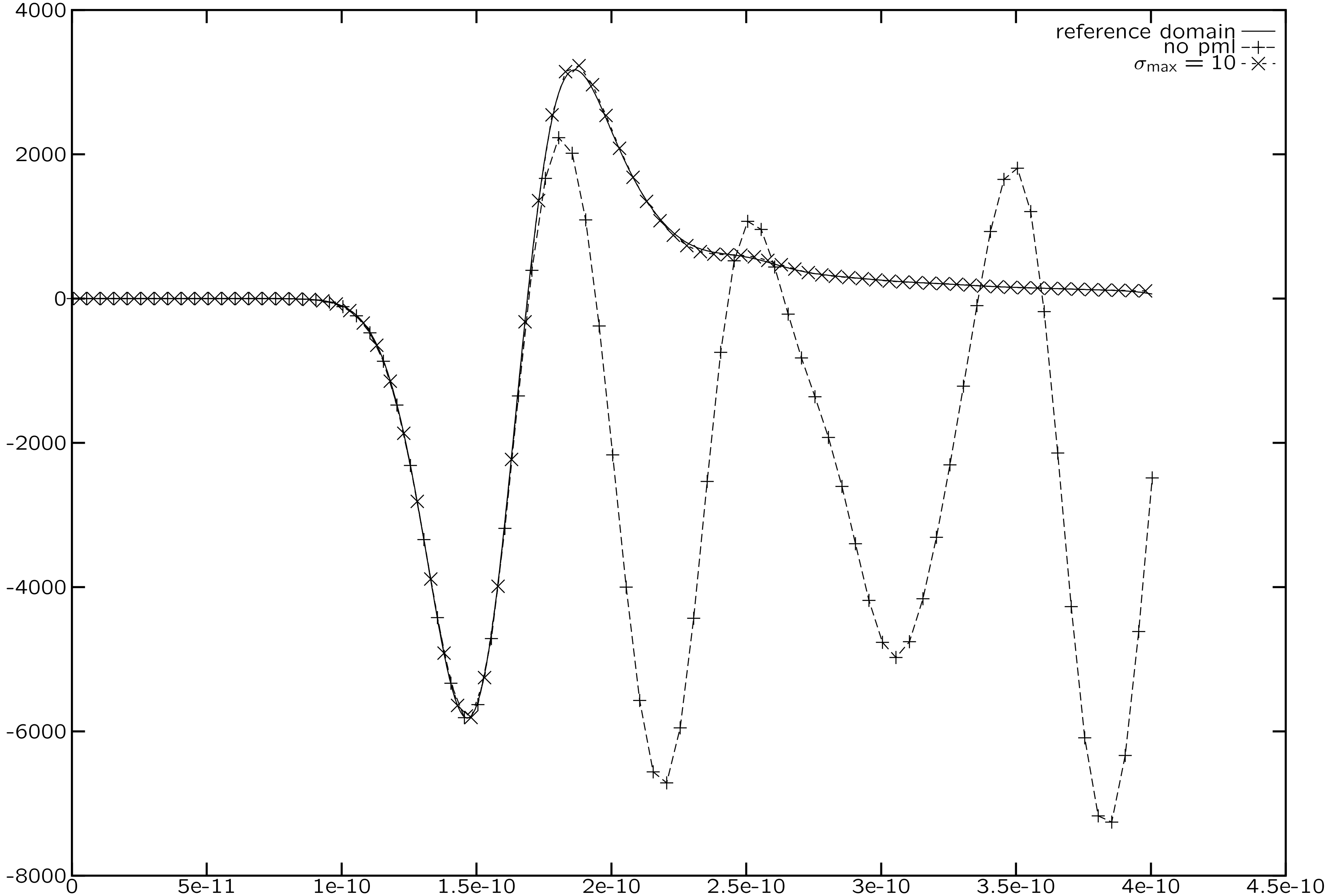


reference
no pml
1.0
10
100
1000
10000
6000
4000
2000
0
-2000
-4000
-6000
-8000
0
5e-11
1e-10
1.5e-10
2e-10
2.5e-10
3e-10
3.5e-10
4e-10
4.5e-10

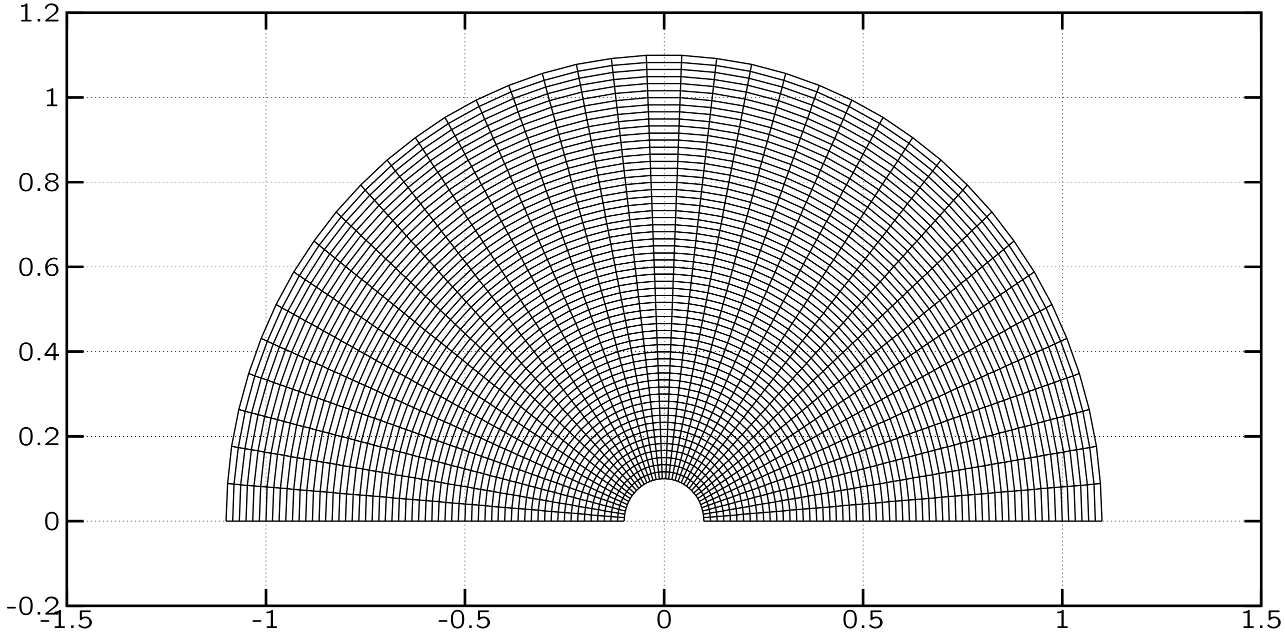
## Locally Non-orthogonal grids

To illustrate the effectiveness of the mimetic operators on non-orthogonal grids.

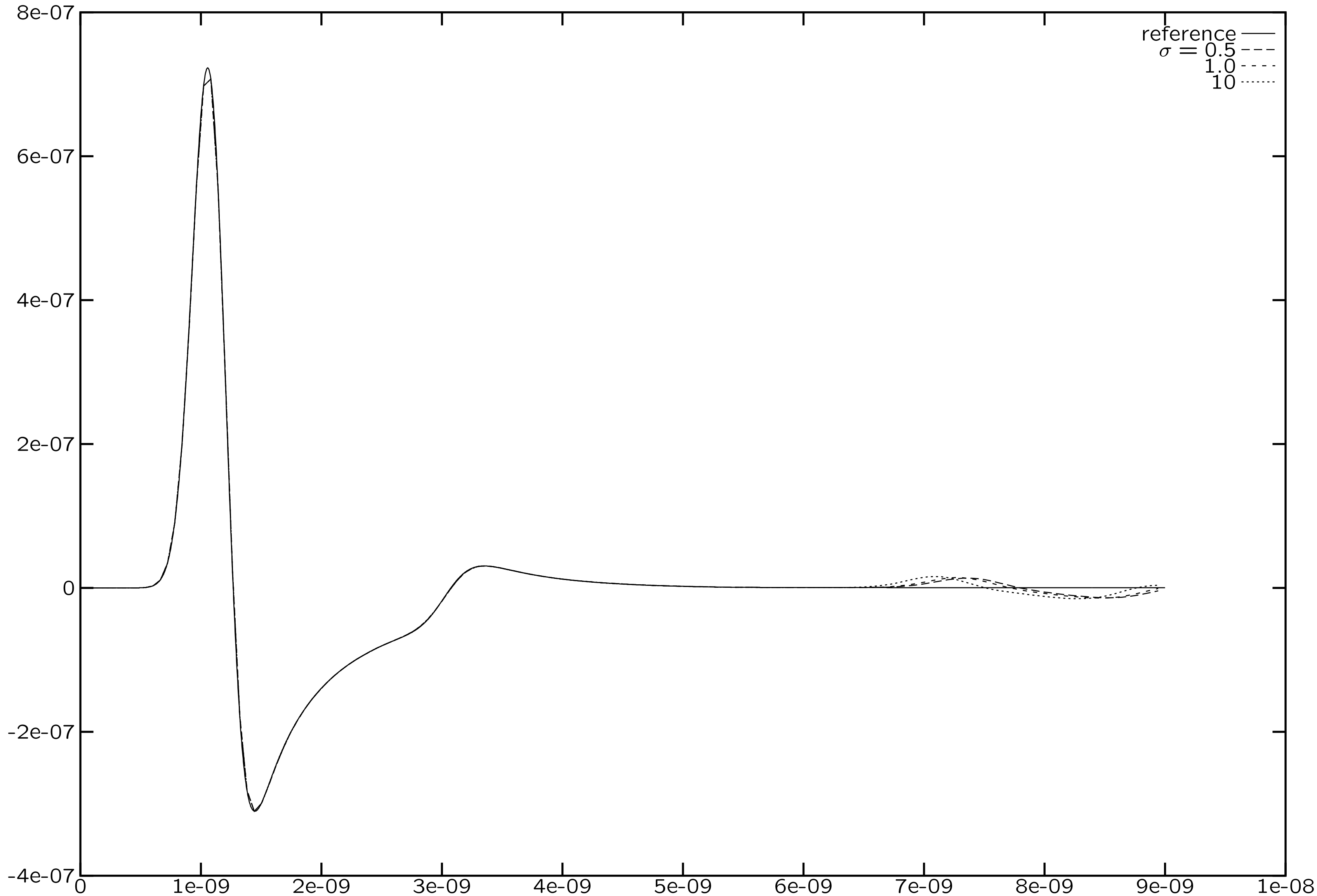



reference domain
no pml
$\sigma_{max} = 10$
4000
2000
0
-2000
-4000
-6000
-8000
0
5e-11
1e-10
1.5e-10
2e-10
2.5e-10
3e-10
3.5e-10
4e-10
4.5e-10

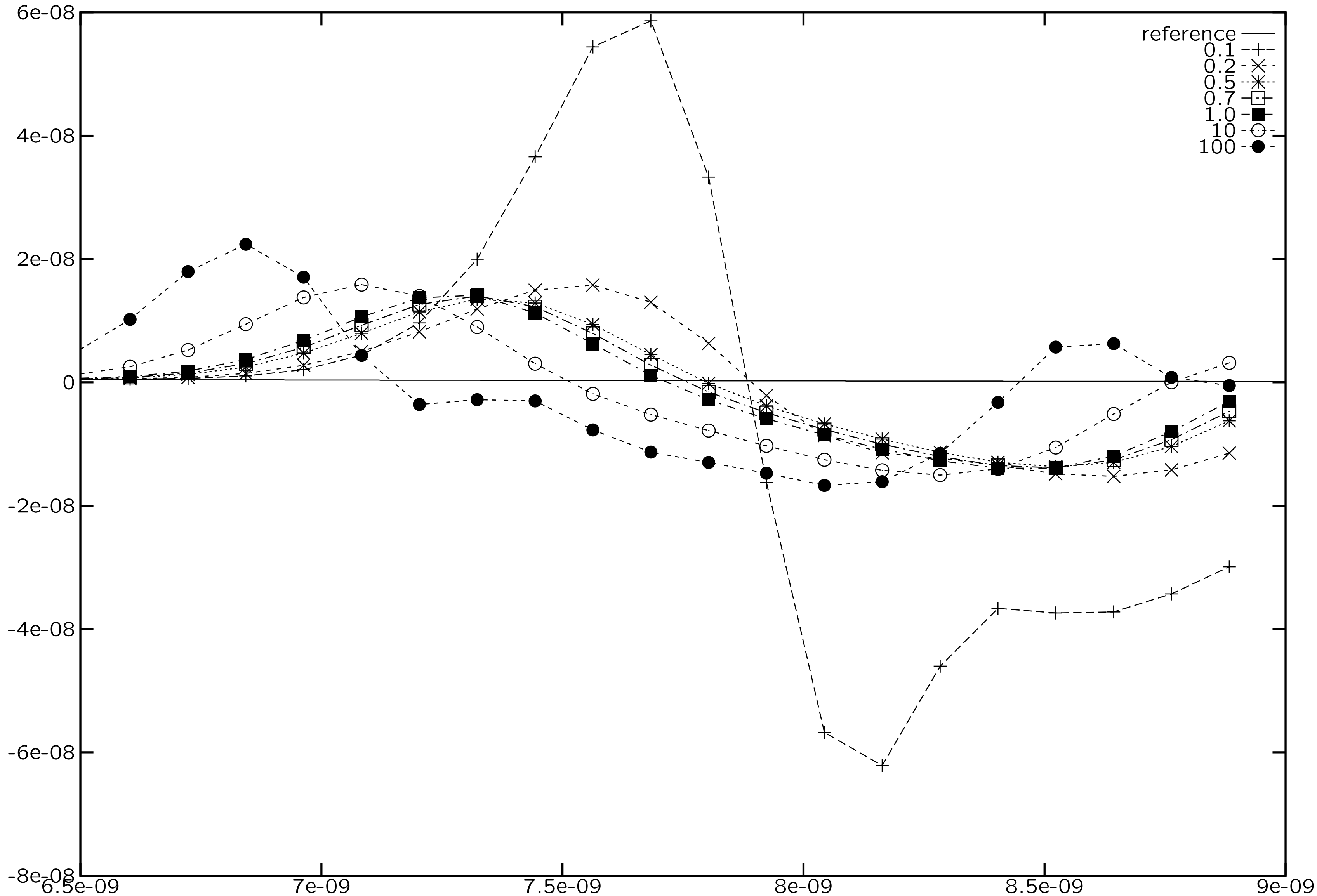
## Polar Grid




Scattered field reflection off conducting cylinder simulated through boundary condition on inner edge. 12 cell PML placed on outer edge. $B_z$ observed at point $(-0.116, 0.009)$.



8e-07
6e-07
4e-07
2e-07
0
-2e-07
-4e-07
0
1e-09
2e-09
3e-09
4e-09
5e-09
6e-09
7e-09
8e-09
9e-09
1e-08
reference
$\sigma = 0.5$
1.0
10



reference
0.1
0.2
0.5
0.7
1.0
10
100
6e-08
4e-08
2e-08
0
-2e-08
-4e-08
-6e-08
-8e-08
6.5e-09
7e-09
7.5e-09
8e-09
8.5e-09
9e-09

Future Work

Re-express curl operators in terms of continuous field quantities. This involves folding the PML tensors into the $\overline{{}_\epsilon\mathbf{curl}_\mu}$ operator.

Implement non-orthogonal PML through a local conversion to an orthogonal basis.